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NUMERICAL BOUNDARY CONDITIONS FOR UNSTEADY TRANSONIC FLOW CALCULATIONS*

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SUMMARY

In calculations of transonic flows it is necessary to limit the domain of computation to a size that is manageable by computers. At the boundary of the computational domain, boundary conditions are required to ensure a unique solution. Since wave solutions exist in the unsteady transonic flow field, incorrect boundary conditions may result in spurious reflections from the computational boundary. This may introduce errors into the solution. To prevent the spurious reflections, absorbing boundary conditions are often used on the computational boundary. In this paper we describe a method to derive absorbing boundary conditions for transonic calculations. We demonstrate both theoretically and numerically that the use of the absorbing boundary conditions will reduce the spurious reflections in the calculation.

KEY WORDS Absorbing boundary conditions Unsteady transonic flows

1. INTRODUCTION

In the calculation of transonic flows the governing equation needs to be solved over the entire flow field. Since the flow field is unbounded, techniques are needed to limit computations to a size that is manageable by computers. One technique is to use co-ordinate transformations, in which the infinite flow field is mapped into a finite computational domain. While this approach is effective for steady problems, it is unsatisfactory for unsteady calculations. An alternative is to truncate the flow field to a finite computational domain by introducing a computational boundary. On the boundary, boundary conditions are needed to ensure a unique solution. Since wave solutions exist in unsteady problems, the interaction of waves with the computational boundary cannot be avoided for long-time calculations. If boundary conditions are incorrect, incident waves may give rise to spurious reflections at the computational boundary. These spurious reflections will introduce a large error in the solution within the computational domain. Therefore boundary conditions that absorb incident waves are needed to prevent the spurious reflections in the solution. Such conditions are said to be absorbing boundary conditions or radiation boundary conditions.

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There is a substantial literature on absorbing boundary conditions and one of the landmark papers is by Engquist and Majda.[1] In Reference 1 absorbing boundary conditions were obtained for the wave equations by considering a pseudodifferential equation. Gustafsson[2] obtained far-field boundary conditions for Euler's equations. Higdon[3] found a general representation of the Engquist–Majda boundary conditions. Absorbing boundary conditions for general second-order hyperbolic equations were considered in Reference 4. More recently, absorbing boundary conditions for dispersive waves were analysed in Reference 5. Most work on absorbing boundary conditions has been concentrated on waves in a homogeneous medium.

The purpose of this paper is to derive absorbing boundary conditions suitable for unsteady transonic calculations and to demonstrate that the use of absorbing boundary conditions will improve the accuracy and efficiency of the solution. Absorbing boundary conditions for transonic flows will be obtained by using the method introduced in Reference 4, where absorbing boundary conditions were obtained for waves propagating in a homogeneous medium. In this paper we will extend the results of Reference 4 to non-homogeneous media.

We assume that the governing equation is the transonic small-disturbance equation given by

$$\frac{\partial f_0}{\partial t} + \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0, \tag{1}$$

where f_i are given in terms of the perturbation potential ϕ by

$$f_0 = -A\phi_t - B\phi_x, \qquad f_1 = E\phi_x + F\phi_x^2 + G\phi_y^2 \qquad f_2 = \phi_y + H\phi_x\phi_y, \qquad f_3 = \phi_z. \tag{2}$$

The coefficients A, B, E, F and G are chosen as 'NASA Ames coefficients'' which are defined as

$$A = M_\infty^2\kappa^2, \qquad B = 2M_\infty^2\kappa, \qquad E = 1 - M_\infty^2,$$
$$F = -\tfrac{1}{2}(\gamma + 1)M_\infty^2, \qquad G = \tfrac{1}{2}(\gamma - 3)M_\infty^2, \qquad H = -(\gamma - 1)M_\infty^2,$$

where κ, M_∞, γ are the reduced frequency, the freestream Mach number and the specific heat ratio respectively. The transonic small-disturbance equation (1) has been widely used for the calculation of transonic flows.[6–8]

The approach used in this paper to derive absorbing boundary conditions for (1) is briefly described as follows. We assume that a computational boundary is place sufficiently far away from the source of disturbances, e.g. the body of an aircraft wing or a shock wave over a wing surface. In the region near the computational boundary it can be assumed that the unsteady flow can be represented as a sum of a steady flow and an unsteady perturbation. Furthermore, the unsteady component is substantially smaller than the steady component in magnitude. Under this assumption the transonic small-disturbance equation (1) can be cast into two equations, one for the steady flow and one for the unsteady perturbation. Once the steady flow potential is obtained from the first equation, the equation for the unsteady perturbation potential becomes linear. This linear equation describes the propagation of unsteady waves in a medium given by the steady state flow field. The propagation velocity of the unsteady waves can then be determined by the geometrical acoustic approximation. Finally, absorbing boundary conditions are obtained by requiring that waves can only propagate from interior to exterior on the boundary. It is worth noting that the present method is different from using the steady state potential as the boundary data. In the present method the solution is still free to oscillate at the boundary, but the medium is assumed to be generated by the steady state flows.

The organization of this paper is as follows. In Section 2 equation (1) is linearized around the steady flow and the geometrical acoustic approximation is used to determine the velocity of wave propagations. In Section 3 we obtain absorbing boundary conditions for unsteady transonic flows and analyse reflection coefficients. Finally, numerical results are presented in Section 4.

2. THE GEOMETRICAL ACOUSTIC APPROXIMATION

Under certain conditions it can be assumed that the unsteady flow potential can be represented as

$$\phi(x, y, z, t) = \phi^{s}(x, y, z) + \varepsilon\phi^{u}(x, y, z, t), \quad \varepsilon \ll 1, \tag{3}$$

where ϕ^{s} is the steady state potential and ϕ^{u} is the unsteady perturbation potential. This assumption applies to the flow around oscillating wings provided that the vibration amplitude is sufficiently small. Tijdeman[9] has found experimentally that this assumption is justified for infinitesimal-amplitude disturbances.

When equation (3) is substituted into (1) and the terms containing ε^2 and higher powers are neglected, the first approximation leads to equations for ϕ^{s} and ϕ^{u} given by

$$\frac{\partial f_1^s}{\partial x} + \frac{\partial f_2^s}{\partial y} + \frac{\partial f_3^s}{\partial z} = 0, \tag{4}$$

where f_i^s are defined by (2) with ϕ replaced by ϕ^{s} and

$$A^{u}\phi_{tt}^{u} + B_{u}\phi_{xt}^{u} = C^{u}\phi_{xx}^{u} + D^{u}\phi_{x}^{u} + E^{u}\phi_{yy}^{u} + F^{u}\phi_{y}^{u} + G^{u}\phi_{xy}^{u} + \phi_{zz}^{u} = 0, \tag{5}$$

where

$$\begin{aligned} &A^{u} = M_{\infty}^{2}\kappa^{2}, \qquad B^{u} = 2M_{\infty}^{2}\kappa, \qquad C^{u} = (1 - M_{\infty}^{2}) - (\gamma + 1)M_{\infty}^{2}\phi_{x}^{s}, \\ &D^{u} = -(\gamma + 1)M_{\infty}^{2}\phi_{xx}^{s} - (\gamma - 1)M_{\infty}^{2}\phi_{yy}^{s}, \\ &E^{u} = 1 - (\gamma - 1)M_{\infty}^{2}\phi_{x}^{s}, \qquad F^{u} = -2M_{\infty}^{2}\phi_{xy}^{s}, \qquad G^{u} = -2M_{\infty}^{2}\phi_{y}^{s}. \end{aligned} \tag{6}$$

Equation (4) is non-linear but time-independent. It can be solved first to determine the coefficients of equation (5). Equation (5) is a linear equation with non-homogeneous coefficients; it describes the propagation of unsteady disturbances in the steady state flow field determined by ϕ^{s}. In the following, since we will only be concerned with the unsteady equation (5), the superscript 'u' will be omitted in the coefficients of equations (5) and (6).

Further approximation can be made by representing the unsteady potential by a local plan wave[10,11]

$$\phi^{u}(x, y, z, t) = a(x, y, z, t)e^{i\theta(x, y, z, t)}. \tag{7}$$

The approximation (7) is valid provided that the flow properties vary with position only gradually on a scale of wavelengths. This allows the waves locally to be approximately sinusoidal, with local wave numbers and frequency given by*

$$\xi = \theta_{x}, \qquad \eta = \theta_{y}, \qquad \zeta = \theta_{z}, \qquad \omega = \theta_{t}. \tag{8}$$

If (7) is substituted into (5) and the terms involving a_x, $\xi_x, \ldots$ and their higher-order derivatives are neglected, a dispersion relation is obtained as

$$A\omega^2 + B\omega\xi = C\xi^2 - iD\xi + E\eta^2 - iF\eta + G\xi\eta + \zeta^2. \tag{9}$$

* The wave numbers are commonly defined by $\xi = -\theta_x$, etc., but for convenience later in defining absorbing boundary conditions, we have omitted the negative sign.

The geometrical acoustics approximation consists of the assumption that the frequency and wave numbers of the unsteady perturbations are very large.[10,11] Thus the terms linear in ξ and η can be neglected in (9). Then the dispersion relation under the geometrical acoustics approximation becomes

$$P(\omega, \xi, \eta, \zeta; x, y, z) = A\omega^2 + B\omega\xi - C\xi^2 - E\eta^2 - G\xi\eta - \zeta^2 = 0, \tag{10}$$

where A, B, C, E and G are given by (6) with superscript 'u' omitted.

The dispersion relation (10) can be solved by the method of characteristics and the solution can be obtained by integrating along its characteristic curves. The characteristics of (10) are defined as curves in the (x, y, z)-space with direction given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = v, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = w, \tag{11}$$

where

$$u = \frac{P_\xi}{P_\omega} = \frac{B\omega - 2C\xi - G\eta}{2A\omega + B\xi}, \qquad v = \frac{P_\eta}{P_\omega} = \frac{2E\eta + G\xi}{2A\omega + B\xi}, \qquad w = \frac{P_\zeta}{P_\omega} = \frac{2\zeta}{2A\omega + B\xi}. \tag{12}$$

The vector $\mathbf{u} = (u, v, w)$ defines the *group velocity*; it is the velocity of energy propagation. Along each path (11) of energy propagation the rates of change of wave numbers and frequency are given by

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\frac{P_x}{P_\omega}, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}t} = -\frac{P_y}{P_\omega}, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}t} = -\frac{P_z}{P_\omega}, \qquad \frac{\mathrm{d}\omega}{\mathrm{d}t} = 0. \tag{13}$$

In other words, (13) specifies the refraction of wave energy.

3. ABSORBING BOUNDARY CONDITIONS

In this section we will derive absorbing boundary conditions. We assume that the computational domain is either a ball or a cube in the (x, y, z)-space. Let the computational boundary be defined by a closed surface Γ which encloses the region of disturbances, such as a wing body or a shock wave. Thus all disturbances are generated in the region interior to Γ. We also assume that in the region exterior to Γ the gradient of steady state potential, $\nabla\phi^s$, varies with position (x, y, z) only gradually on the scale of wavelength. Since the coefficients A, B, C, E and G in $P(\omega, \xi, \eta, \zeta; x, y, z)$ depend only on $\nabla\phi^s$, in the region exterior to Γ the magnitudes of P_x, P_y and P_z are small compared with that of P_ω. It follows from (13) that in the exterior region of Γ the refraction of wave numbers is small, so that a wave only slightly changes its direction of propagation in the exterior region.

We further assume that if a disturbance from the interior, e.g. a disturbance generated by an oscillating wing body or a shock wave, comes to the boundary Γ, it travels in a direction pointing to the exterior of Γ. Denote by $\mathbf{n} = \mathbf{n}(x, y, z)$ the outward normal of Γ at (x, y, z). Then the group velocity $\mathbf{u} = (u, v, w)$ of the disturbance has a positive component in the direction of $\mathbf{n}$. This implies that any disturbance from the interior satisfies

$$\mathbf{u}\cdot\mathbf{n} \geqslant 0 \quad \text{on } \Gamma \tag{14}$$

when it first arrives at the boundary Γ.

The condition (14) is therefore a boundary condition which is satisfied by all incident waves from the interior of Γ. If this condition is imposed on the boundary, all incident waves will be absorbed and no reflections will be generated. For this reason the condition (14) is referred to as a *perfectly absorbing* boundary condition. It is clear that the condition (14) is equivalent to

$$\mathbf{u}\cdot\mathbf{n} - |\mathbf{u}\cdot\mathbf{n}| = 0 \quad \text{on } \Gamma. \tag{15}$$

This expression can be regarded as the dispersion relation of a boundary operator in the physical (x, y, z)-space. Unfortunately, this operator is a pseudodifferential operator which is non-local in both space and time. For computational efficiency a boundary condition involving a differential operator is preferred. Such a condition can be found by using some approximation of (15), after which the boundary condition will no longer be perfectly absorbing but will become partially absorbing.

For an approximation of (15) we consider waves whose energy propagates at a fixed speed $\mu > 0$ in the normal direction $\mathbf{n}$. These waves satisfy the relation

$$\mathbf{u}\cdot\mathbf{n} = \mu \quad \text{on } \Gamma. \tag{16}$$

If we write $\mathbf{n} = (n_x, n_y, n_z)$, then it follows from the definition of the group velocity $\mathbf{u}$ in (12) that (16) can be written as

$$\mathscr{B}(\omega, \xi, \eta, \zeta) \equiv n_x P_\xi + n_y P_\eta + n_z P_\zeta - \mu P_\omega = 0, \tag{17}$$

or, by the definition of P in (10),

$$\mathscr{B}(\omega, \xi, \eta, \zeta) \equiv n_x(B\omega - 2C\xi - G\eta) - n_y(2E\eta + G\xi) - n_z(2\zeta) - \mu(2A\omega + B\xi) = 0. \tag{18}$$

It is clear that (18) is the dispersion relation of a boundary condition of the form

$$n_x(B\phi_t^u - 2C\phi_x^u - G\phi_y^u) - n_y(2E\phi_y^u + G\phi_x^u) - n_z(2\phi_z^u) - \mu(2A\phi_t^u + B\phi_x^u) = 0 \quad \text{on } \Gamma. \tag{19}$$

The boundary condition (19) is a local condition and is easy to implement. However, it is not perfectly absorbing. For a given value of $\mu > 0$ the condition (19) is perfectly absorbing only for those disturbances whose propagation speed in the normal direction $\mathbf{n}$ is μ; no reflection will be generated for such disturbances. For waves of other speeds reflections may still exist, but we will show that the reflections will be small compared with the incident waves.

To see how much reflection may be generated when the boundary condition (19) is used, we consider reflection coefficients. Let

$$\phi^u = e^{i\theta_i} + re^{i\theta_r}, \tag{20}$$

where the first term represents an incident wave and the second term represents the reflected wave; r is defined as the reflection coefficient and is the amplitude of the reflected wave if the incident wave has unit amplitude. A small value of r implies that the amplitude of the reflected wave is small.

Substituting (20) into (19) and solving for r, we find that the reflection coefficient is given by

$$r = \frac{\mathscr{B}(\omega, \xi_i, \eta_i, \zeta_i)}{\mathscr{B}(\omega, \xi_r, \eta_r, \zeta_r)},$$

where $\mathscr{B}$ is as defined in (17) and (ξ_i, η_i, ζ_i) and (ξ_r, η_r, ζ_r) are wave vectors of the incident and reflected waves respectively. If we write $\nabla_{(\xi,\eta,\zeta)}P = (P_\xi, P_\eta, P_\zeta)$, then the reflection coefficient becomes

$$r = \frac{(\nabla_{(\xi,\eta,\zeta)}P)_i \cdot \mathbf{n} - \mu(P_\omega)_i}{(\nabla_{(\xi,\eta,\zeta)}P)_r \cdot \mathbf{n} - \mu(P_\omega)_r} \tag{21}$$

where $(\cdot)_i$ and $(\cdot)_r$ denote evaluations at (ξ_i, η_i, ζ_i) and (ξ_r, η_r, ζ_r) respectively.

Given a wave vector of an incident wave (ξ_i, η_i, ζ_i), the wave vector (ξ_r, η_r, ζ_r) of the reflected wave can be determined as follows. First, since both the incident and reflected waves have the same phase on the boundary, we have $\theta_i = \theta_r$. This implies that

$$(\xi_i, \eta_i, \zeta_i) \cdot \boldsymbol{\tau} = (\xi_r, \eta_r, \zeta_r) \cdot \boldsymbol{\tau},$$

where $\boldsymbol{\tau}$ is any unit tangential vector of the boundary surface Γ. Next, both (ξ_i, η_i, ζ_i) and (ξ_r, η_r, ζ_r) satisfy the dispersion relation (10), so that in the (ξ, η, ζ)-space both (ξ_i, η_i, ζ_i) and (ξ_r, η_r, ζ_r) lie on the ellipsoid defined by (10). Now, to find (ξ_r, η_r, ζ_r), we draw in the (ξ, η, ζ)-space a straight line at the point (ξ_i, η_i, ζ_i) in the direction of $\mathbf{n}$. The line intersects the ellipsoid defined by (10) at two points. One is (ξ_i, η_i, ζ_i) and the other is (ξ_r, η_r, ζ_r). By this construction we can easily verify that

$$(\nabla_{(\xi,\eta,\zeta)}P)_r \cdot \mathbf{n} = -(\nabla_{(\xi,\eta,\zeta)}P)_i \cdot \mathbf{n}. \tag{22}$$

From (21) the reflection coefficient can now be written as

$$r = \frac{(\nabla_{(\xi,\eta,\zeta)}P)_i \cdot \mathbf{n}}{(\nabla_{(\xi,\eta,\zeta)}P)_r \cdot \mathbf{n}} \frac{1 - \mu(P_\omega)_i/[(\nabla_{(\xi,\eta,\zeta)}P)_i \cdot \mathbf{n}]}{1 - \mu(P_\omega)_r/[(\nabla_{(\xi,\eta,\zeta)}P)_r \cdot \mathbf{n}]},$$

and, owing to (22), we have

$$|r| = \left| \frac{1/\mu - (P_\omega)_i/[(\nabla_{(\xi,\eta,\zeta)}P)_i \cdot \mathbf{n}]}{1/\mu - (P_\omega)_r/[(\nabla_{(\xi,\eta,\zeta)}P)_r \cdot \mathbf{n}]} \right|. \tag{23}$$

The expression in (23) can be simplified if we observe that

$$\frac{(P_\omega)_i}{(\nabla_{(\xi,\eta,\zeta)}P)_i \cdot \mathbf{n}} = \left[\left(\frac{\nabla_{(\xi,\eta,\zeta)}P}{P_\omega} \right)_i \cdot \mathbf{n} \right]^{-1} = (\mathbf{u}_i \cdot \mathbf{n})^{-1}$$

and

$$\frac{(P_\omega)_r}{(\nabla_{(\xi,\eta,\zeta)}P)_r \cdot \mathbf{n}} = \left[\left(\frac{\nabla_{(\xi,\eta,\zeta)}P}{P_\omega} \right)_r \cdot \mathbf{n} \right]^{-1} = (\mathbf{u}_r \cdot \mathbf{n})^{-1}.$$

Now the reflection coefficient can be written as

$$|r| = \left| \frac{1/\mu - (\mathbf{u}_i \cdot \mathbf{n})^{-1}}{1/\mu - (\mathbf{u}_r \cdot \mathbf{n})^{-1}} \right|. \tag{24}$$

The relation (24) shows that the reflection coefficient is zero if the incident wave satisfies $\mathbf{u}_i \cdot \mathbf{n} = \mu$, i.e. the condition (19) is perfectly absorbing for such waves. In general the reflection coefficients is non-zero. However, by choosing a proper μ, the reflection coefficient can be shown to be less than unity for all incident waves that do not travel too slowly. Indeed, for any $\varepsilon > 0$ we can always find a $\mu > 0$ such that $|r| < 1$ for all waves satisfying $\mathbf{u}_i \cdot \mathbf{n} \geqslant \varepsilon$. This shows that the boundary condition (19) is absorbing for all waves except the slowly travelling waves, i.e. those waves with $\mathbf{u}_i \cdot \mathbf{n} \in [0, \varepsilon)$.

We note that the reflection coefficient for the perfectly absorbing boundary condition (15) is zero for all incident waves. It can also be shown that the reflection coefficient of the boundary condition $\phi^u = 0$ is always unity. For this reason the condition $\phi^u = 0$ is often said to be a totally reflecting boundary condition.

The absorptivity of (19) can be improved by adjusting the parameter μ. For example, if one has *a priori* knowledge of the velocity of disturbances arriving at the boundary, the parameter μ may be tuned to this velocity. If no information is known about the solution, μ can be selected to annihilate the fastest-propagating waves, i.e. set $\mu = \max|\mathbf{u}_i \cdot \mathbf{n}|$. The maximum group velocity $\max|\mathbf{u}_i \cdot \mathbf{n}|$ can be obtained analytically from (12) without *a priori* knowledge of the solution. It has been observed in numerical experiments that the solution is not sensitive to μ. This fact is also reported in Reference 5. Furthermore, the parameter μ may also vary with (x, y, z) on the computational boundary Γ provided that it is slowly varying in the scale of wavelengths of disturbance waves.

Another method to reduce reflections is to use an operator consisting of many factors of (18). The dispersion relation of a condition consisting of two factors, for example, is given by

$$\mathscr{B}(\omega, \xi, \eta, \zeta) = (\nabla_{(\xi,\eta,\zeta)} P \cdot \mathbf{n} - \mu_1 P_\omega)(\nabla_{(\xi,\eta,\zeta)} P \cdot \mathbf{n} - \mu_2 P_\omega) = 0, \quad \mu_1, \mu_2 > 0. \tag{25}$$

The parameters μ_1 and μ_2 can be tuned to waves with specific velocities. The reflection coefficient for (25) can be found to be

$$|r| = \left|\frac{1/\mu_1 - (\mathbf{u}_i \cdot \mathbf{n})^{-1}}{1/\mu_1 - (\mathbf{u}_r \cdot \mathbf{n})^{-1}}\right| \left|\frac{1/\mu_2 - (\mathbf{u}_i \cdot \mathbf{n})^{-1}}{1/\mu_2 - (\mathbf{u}_r \cdot \mathbf{n})^{-1}}\right|. \tag{26}$$

Since each factor in (26) is smaller than unity, the amplitude of the reflection when (25) is used is smaller than that when (18) is used. For more details on higher-order boundary conditions we refer to References 3–5.

4. NUMERICAL RESULTS

The above absorbing boundary conditions have been implemented in a three-dimensional transonic simulation code UST3D.[8]

In UST3D the computational domain is a region as shown in Figure 1. The computational boundaries are planes normal to the co-ordinate axes. Thus the upstream boundary is

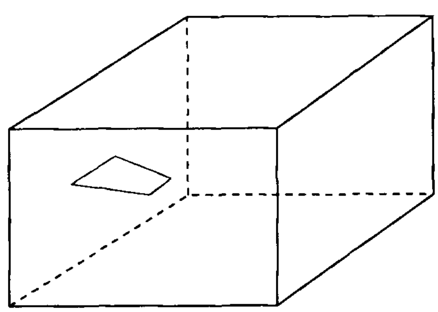

Figure 1. The computational region

given by $x = -l$. On this boundary the dispersion relation (18) of an absorbing boundary condition is

$$\alpha\omega + \beta\xi + \gamma\eta + 0, \tag{27}$$

where

$$\alpha = \sqrt{\left(\frac{B^2}{4} + AC\right)} - \frac{B}{2} - \left(\frac{1}{\mu} + 1\right)\frac{2AC}{\sqrt{(B^2 + 4AC)}},$$

$$\beta = A\left[\left(1 + \frac{1}{\mu}\right)\frac{-B}{\mu\sqrt{(B^2 + 4AC)}} - 1\right], \qquad \gamma = \frac{-AG}{\mu\sqrt{(B^2 + 4AC)}},$$

with $\mu > 0$. From (27) the boundary condition can be derived as

$$\alpha\phi^{\mathrm{u}}_t + \beta\phi^{\mathrm{u}}_x + \gamma\phi^{\mathrm{u}}_y = 0 \quad \text{on } x = -l. \tag{28}$$

The condition (28) gives an exact condition for a plane wave solution if its upstream propagation speed is μ. If the condition (28) is used at the upstream boundary, then plane waves which travel at an upstream μ will be annihilated exactly, so that no reflection will be generated. However, disturbances in general consist of waves of different propagating speeds. Although the boundary condition (28) does not absorb all waves totally, it helps to reduce reflections generated by waves of other propagating speeds; see (24). In the computations the parameter μ was tuned to the fastest group velocity as obtained from (12). It is worth noting that we have tested different values of μ and the solutions are not sensitive to it. The results presented in this paper are typical of the cases we have tested. The downstream boundary condition has the same form as (28) with μ replaced by $-\mu$. Absorbing boundary conditions for other boundaries can be similarly derived. For example, on the boundaries above or below the wing in the z-direction ($z = h$ or $z = -h$) the absorbing boundary conditions are given by

$$\phi^{\mathrm{u}}_z + \mu\sqrt{A}\phi^{\mathrm{u}}_t + \frac{B\mu}{2\sqrt{A}}\phi^{\mathrm{u}}_x = 0, \tag{29}$$

where $0 < \mu \leqslant 1$ on $z = h$ and $-1 \leqslant \mu < 0$ on $z = -h$.

In our computations UST3D was used to compute the transonic flows about an isolated wing oscillating in pitching mode. The wing is divided into many spanwise sections and the pressure distribution at each section is computed. The pressure is represented by the real (RE) and imaginary (Im) parts which are defined by

$$C_{\mathrm{p}} = \mathrm{Re}\sin(\kappa t) - \mathrm{Im}\cos(\kappa t).$$

To evaluate the performance of the absorbing boundary conditions, we may compare the results computed using the absorbing boundary conditions with the exact solution. However, the exact solution is generally unknown. One approach is to use some experimental result as the exact solution. We choose not to use this approach, because any disagreement between the computed results and the experimental result may also be due to limitations of the transonic small-disturbance model. In order to find the error purely due to the use of boundary conditions, we will use the solution on a very large computational domain as the exact solution.

We compare the pressure distributions computed by using different domain sizes and boundary conditions. The boundary conditions compared are the absorbing conditions of this paper and the boundary condition $\phi^{\mathrm{u}} = 0$.[6] The latter boundary condition is totally reflecting, so that outgoing waves will be totally reflected upon hitting the boundaries. The reflections will

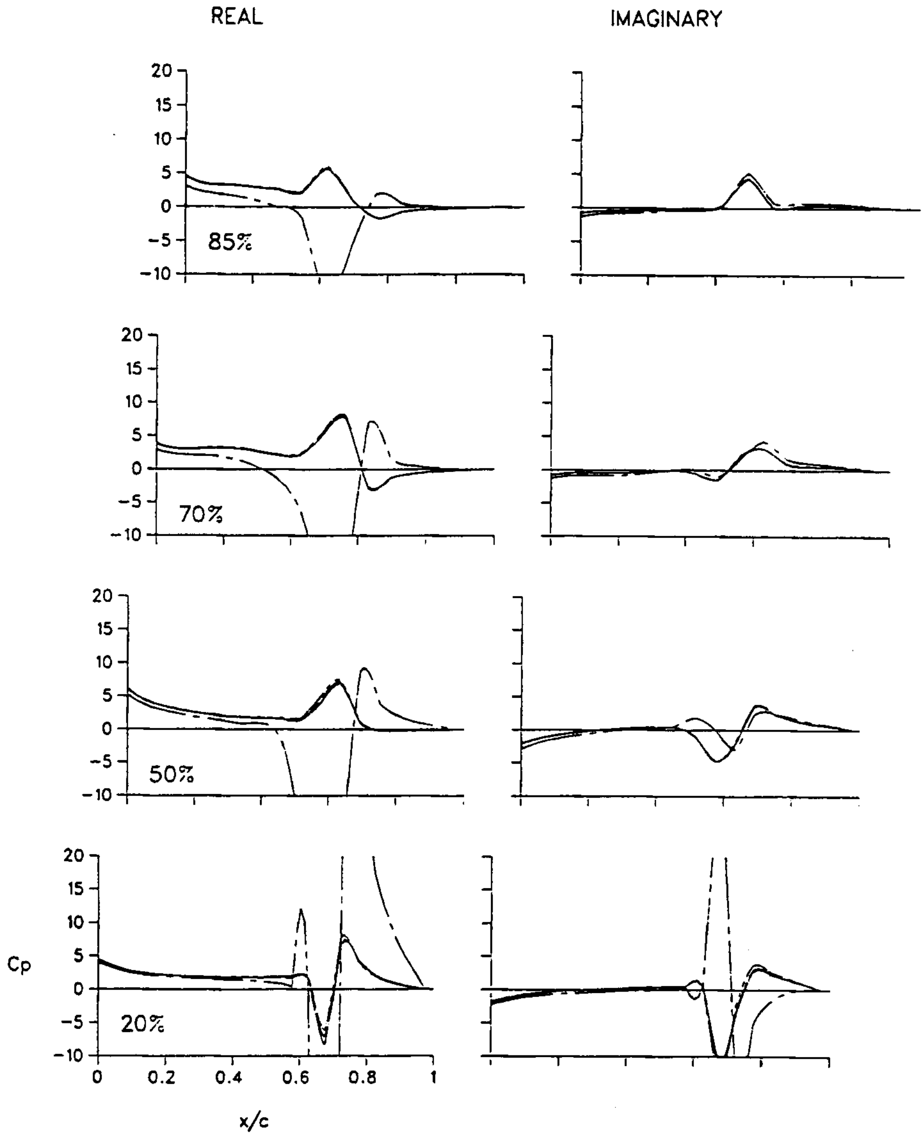

Figure 2. Results for the ONERA M6 wing: ——, (29), larger domain; – – – —, (29) smaller domain; — — —, $\phi^u = 0$, larger domain; —— - ——, $\phi^u = 0$, smaller domain

come back to the domain of computation to contaminate the solution. To prevent the reflection from contaminating the solution, boundaries must be placed for away from the wing, so that during the time period of computation the reflection has not travelled far enough to contaminate the solution at the wing surface where the pressure distributions are calculated. If boundaries are too close, then the boundary condition $\phi^u = 0$ will give rise to a large error in the solution. On the other hand, the absorbing boundary conditions permit boundaries to be place close to the wing surface. In the following we will compare the effect of boundary conditions in the z-direction only.

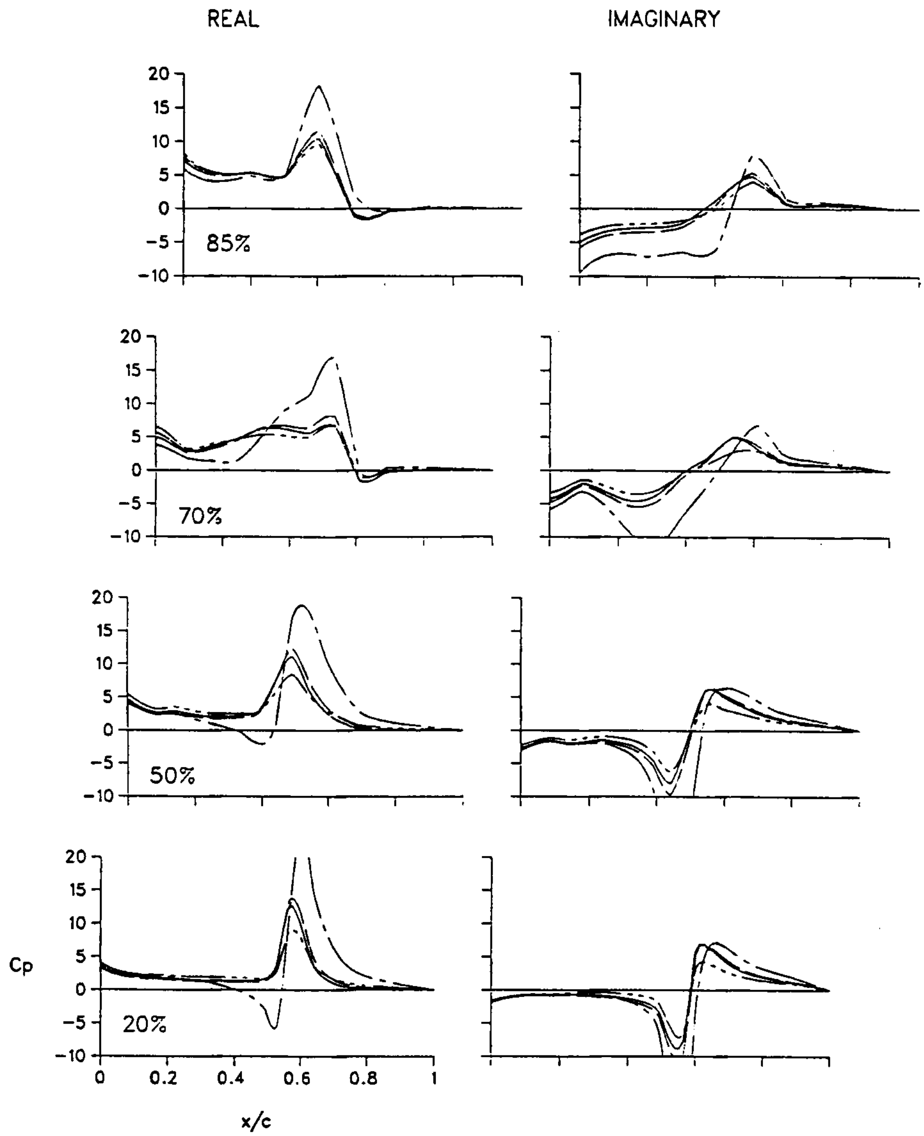

Figure 3. Results for the F5 wing: same curves as Figure 2

Four results are presented for each wing configuration. They are obtained by using two types of boundary conditions on the boundaries of two domains. The boundary of the larger domain is placed at about 7 chord lengths above the wing, while the boundary of the smaller domain is about 0·6 chord lengths above the wing. Since the boundary of the larger domain is far away, the solution obtained using either boundary condition may be considered as an accurate solution. However, the solutions on the smaller domain may be subject to contamination from reflections. Therefore the difference between the solutions on the two domains is a measure of how good a boundary condition is.

The computed pressure distributions at different spanwise locations for ONERA M6 and F5 wings are presented in Figures 2 and 3 respectively. In both cases the Mach number is $M_\infty = 0{\cdot}9$, the reduced frequency is $\kappa = 0{\cdot}55$ and the angle of pitching oscillation is $\alpha = 1°$. In each graph the three curves which are close together represent the results for both types of boundary conditions on the larger domain and for the absorbing boundary condition on the smaller domain. The results computed using the boundary condition $\phi^{u} = 0$ on the smaller domain are represented by the fourth curve, which has a completely different behaviour compared with the other three. These results show that the use of absorbing boundary conditions can significantly improve the accuracy of solutions when a small computational domain is used.

5. CONCLUSIONS

We have derived absorbing boundary conditions for computing transonic flows. The conditions are obtained by requiring that at the computational boundary, disturbance waves travel in the outward direction only. The conditions obtained are perfectly absorbing for waves travelling at a certain speed. The analysis of the reflection coefficient shows that although reflection may exist in general, the amplitude of the reflection is small compared with that of the incident waves. Numerical results are presented to demonstrate that the use of absorbing boundary conditions can improve the accuracy of solutions.

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